

RANDOM MULTIPLICATIVE FUNCTIONS IN SHORT INTERVALS

SOURAV CHATTERJEE AND KANNAN SOUNDARARAJAN

ABSTRACT. We consider random multiplicative functions taking the values ± 1 . Using Stein's method for normal approximation, we prove a central limit theorem for the sum of such multiplicative functions in appropriate short intervals.

1. INTRODUCTION

Many of the functions of interest to number theorists are multiplicative. That is they satisfy $f(mn) = f(m)f(n)$ for all coprime natural numbers m and n . Some examples are the Möbius function $\mu(n)$, the function n^{it} for a real number t , and Dirichlet characters $\chi(n)$. Often one is interested in the behavior of partial sums $\sum_{n \leq x} f(n)$ of such multiplicative functions. For the proto-typical examples mentioned above it is a difficult problem to obtain a good understanding of such partial sums. A guiding principle that has emerged is that partial sums of specific multiplicative functions (e.g. characters or the Möbius function) behave like partial sums of random multiplicative functions. By random we mean that the values of the multiplicative function at primes are chosen randomly, and the values at all natural numbers are built out of the values at primes by the multiplicative property. For example this viewpoint is explored in the context of finding large character sums in [4].

This raises the question of the distribution of partial sums of random multiplicative functions, and even this model problem appears difficult to resolve. The aim of this paper is to study the distribution of random multiplicative functions in short intervals $[x, x+y]$, and in suitable ranges we shall establish that the sum of a random multiplicative function in that range has an approximately Gaussian distribution.

Throughout p will denote a prime number, and let $X(p)$ denote independent random variables taking the values $+1$ or -1 with equal

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probability. Let $X(n) = 0$ if n is divisible by the square of any prime, and if $n = p_1 \cdots p_k$ is square-free we define $X(n) = \prod_{j=1}^k X(p_j)$. Let $M(x) = \sum_{n \leq x} X(n)$. In [5], Halasz showed that with probability 1 we have

$$|M(x)| \leq cx^{\frac{1}{2}} \exp\left(d(\log \log x \log \log \log x)^{\frac{1}{2}}\right),$$

for some positive constants c (which may depend on random function X) and d (an absolute constant), and forthcoming work of Lau, Tenenbaum and Wu [10] substantially improves upon this bound. Furthermore, Halasz showed that with positive probability the estimate $M(x) \geq cx^{\frac{1}{2}} \exp(-d(\log \log x \log \log \log x)^{\frac{1}{2}})$ holds infinitely often (for any $d > 0$), and this has been substantially improved in forthcoming work of Harper [7]. These results may be seen as approximations to the law of the iterated logarithm for sums of independent random variables. In related recent works Hough [8] and Harper [6] have considered the distribution of $\sum'_{n \leq x} X(n)$, where the sum is restricted to integers having exactly k prime factors. Note that the central limit theorem covers the case $k = 1$ when we have a sum of independent random variables. When k is a fixed positive integer, using the method of moments Hough established that such sums have a Gaussian distribution. The work of Harper extends Hough's result and using the martingale central limit theorem he established that the Gaussian distribution persists for $k = o(\log \log x)$, and fails for k of size a constant times $\log \log x$. Recall that most numbers $n \leq x$ have about $\log \log x$ prime factors, and so the dichotomy seen in Harper's result is quite interesting. Harper also showed by a conditioning argument that $M(x)$ itself cannot have a normal distribution with mean 0 and variance the number of square-free integers below x .

Theorem 1.1. *Let X denote a random multiplicative function as above. Let x and y be large natural numbers with $y = \delta x$ for some $\delta < 1/10$. Let $S = S(x, y)$ denote the number of square-free integers in $[x, x + y]$. Let Z denote a Gaussian random variable with mean 0 and variance 1, and let ϕ denote a Lipschitz function satisfying $|\phi(\alpha) - \phi(\beta)| \leq |\alpha - \beta|$ for all real numbers α and β . Then we have that*

$$\left| \mathbb{E} \left(\phi \left(\frac{1}{\sqrt{S}} \sum_{x < n \leq x+y} X(n) \right) \right) - \mathbb{E} \phi(Z) \right|$$

is bounded by a constant times

$$\min \left(1, \left(\frac{y}{S} \right)^{\frac{3}{2}} \frac{1}{(\log 1/\delta)^{\frac{1}{2}}} + \frac{y}{S} \sqrt{\delta \log x} + \frac{y \log x}{S^{\frac{3}{2}} \log y} \right).$$

We recall that the Kantorovich-Wasserstein distance between two probability measures μ and ν on the real line, denoted $\mathcal{W}(\mu, \nu)$, is defined as the supremum of $|\int h d\mu - \int h d\nu|$ over all Lipschitz functions h satisfying $|h(\alpha) - h(\beta)| \leq |\alpha - \beta|$ for all real numbers α and β . Thus our Theorem gives an estimate for the Kantorovich-Wasserstein distance between a normal distribution with mean zero and variance 1, and the distribution of sums of random multiplicative functions in short intervals. An intuitive way to assess the distance between two probability measures is the Kolmogorov statistic: $\mathcal{K}(\mu, \nu) = \sup_{x \in \mathbb{R}} |\int_{-\infty}^x d\mu - \int_{-\infty}^x d\nu|$. By a standard smoothing argument, we shall show how our estimate for the Kantorovich-Wasserstein distance can be used to bound the Kolmogorov statistic.

Corollary 1.2. *With notations as in Theorem 1.1 we have that*

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P} \left(\frac{1}{\sqrt{S}} \sum_{x < n \leq x+y} X(n) \in (-\infty, t) \right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-z^2/2} dz \right|$$

is bounded by a constant times

$$\min \left(1, \left(\frac{y}{S} \right)^{\frac{3}{4}} \frac{1}{(\log 1/\delta)^{\frac{1}{4}}} + \left(\frac{y}{S} \right)^{\frac{1}{2}} (\delta \log x)^{\frac{1}{4}} + \frac{\sqrt{y \log x}}{S^{\frac{3}{4}} \sqrt{\log y}} \right).$$

In an interval $[x, x+y]$ we expect that there are about $\sim \frac{6}{\pi^2}y$ square-free integers. The work of Filaseta and Trifonov [2] shows that if $x \geq y \geq Cx^{\frac{1}{5}} \log x$ for some positive constant C then a positive proportion of the integers in $[x, x+y]$ are square-free. The theorem in Filaseta and Trifonov only asserts the existence of a square-free integer in such an interval, but their proof plainly gives the stronger result above. Therefore for all short intervals with $Cx^{\frac{1}{5}} \log x < y = o(x/\log x)$, our Theorem shows that the distribution of $\sum_{x < n \leq x+y} X(n)$ is approximately normal. Granville [3] has shown that the *ABC*-conjecture implies that the interval $[x, x+y]$ contains a positive proportion of square-free integers if $x^\epsilon \ll y \leq x$ for any $\epsilon > 0$; again Granville only stated the existence of one square-free integer in such intervals, but his proof gives the stronger assertion above. Thus, on the *ABC*-conjecture, for any short interval with $x^\epsilon \ll y = o(x/\log x)$ our Theorem shows that the distribution of $\sum_{x < n \leq x+y} X(n)$ is approximately normal.

The proof of this result is based on a version of Stein's method for normal approximation developed in [1]. This involves calculating quantities related to the fourth moment of $\sum_{x < n \leq x+y} X(n)$. The fourth moment itself is calculated in Proposition 3.1 below. If the interval $[x, x+y]$ contains a positive proportion of square-free numbers, then Proposition 3.1 shows that the fourth moment is asymptotically the

fourth moment of a normal distribution provided $y = o(x/\log x)$. Further, when y is of size a constant times $x/\log x$, the argument there shows that the fourth moment does not match the fourth moment of a normal distribution. Thus it seems plausible that for $x/\log x \ll y \leq x$ the distribution of $\sum_{x < n \leq x+y} X(n)$ is not normal, but we do not have a proof of this assertion. By modifying the conditioning argument in Harper [5] we can establish that if y is of a constant times x then the distribution of $\sum_{x < n \leq x+y} X(n)$ is not normal.

The method developed here could also be used to study the distribution of $\sum_{n \in \mathcal{S}} X(n)$ for other subsets \mathcal{S} of square-free numbers in $[1, x]$. For example, we can obtain in this manner a different treatment of the results of Harper and Hough. Another example is the set of integers below x that are $\equiv a \pmod{q}$ where $(a, q) = 1$. If $q/\log x$ is large, and this arithmetic progression contains the expected number of square-free integers, the distribution should be normal analogously to Theorem 1.1.

2. BEGINNING OF THE PROOF

Let x , y and δ be as in the statement of the Theorem, and let X denote a random multiplicative function as defined in the Introduction. We let z denote $\frac{1}{2} \log(1/\delta)$. We divide the primes below $2x$ into large (that is $> z$) and small (that is $\leq z$) primes. We denote the set of large primes by \mathcal{L} , and the set of small primes by \mathcal{S} . Let \mathcal{F} be the sigma-algebra generated by $X(p)$ for all $p \in \mathcal{S}$, and we denote the conditional expectation given \mathcal{F} by $\mathbb{E}^{\mathcal{F}}$.

Let $X_{\mathcal{L}}$ denote the vector $(X(p))_{p \in \mathcal{L}}$. Then, given \mathcal{F} , we may think of $\sum_{x < n \leq x+y} X(n)$ as a function of $X_{\mathcal{L}}$, and we write this function as $f(X_{\mathcal{L}})$.

Lemma 2.1. *With the above notations we have*

$$\mathbb{E}^{\mathcal{F}}(f(X_{\mathcal{L}})) = 0,$$

and

$$\mathbb{E}^{\mathcal{F}}(f(X_{\mathcal{L}})^2) = S(x, y).$$

Proof. Write a square-free number $n \in [x, x+y]$ as $n_{\mathcal{S}} n_{\mathcal{L}}$ where $n_{\mathcal{S}}$ is the product of the primes in \mathcal{S} that divide n , and $n_{\mathcal{L}}$ the product of the primes in \mathcal{L} that divide n . From our choice of $z = \frac{1}{2} \log(1/\delta)$ we note that $n_{\mathcal{S}} \leq \prod_{p \leq z} p \leq 4^z$. It follows that $n_{\mathcal{L}} = n/n_{\mathcal{S}} > \delta x = y$. From this we obtain that $\mathbb{E}^{\mathcal{F}}(f(X_{\mathcal{L}})) = 0$. Moreover, note that if n and n' are distinct square-free numbers in $[x, x+y]$ then we must have $n_{\mathcal{L}} \neq n'_{\mathcal{L}}$. Therefore we deduce that $\mathbb{E}^{\mathcal{F}}(f(X_{\mathcal{L}})^2) = S(x, y)$, proving our Lemma. \square

Let $X'_\mathcal{L}$ denote an independent copy of $X_\mathcal{L}$. For each subset \mathcal{A} of \mathcal{L} we write $X_\mathcal{L}^\mathcal{A}$ to be the vector defined as $X^\mathcal{A}(p) = X(p)$ for $p \in \mathcal{L} \setminus \mathcal{A}$, and $X^\mathcal{A}(p) = X'(p)$ for $p \in \mathcal{A}$. For a proper subset \mathcal{A} of \mathcal{L} , and a prime $p \in \mathcal{L} \setminus \mathcal{A}$ we define

$$\Delta_p f := f(X_\mathcal{L}) - f(X_\mathcal{L}^{\{p\}}),$$

and

$$\Delta_p f^\mathcal{A} := f(X_\mathcal{L}^\mathcal{A}) - f(X_\mathcal{L}^{\mathcal{A} \cup \{p\}}).$$

Finally define

$$T := \frac{1}{2} \sum_{\mathcal{A} \subsetneq \mathcal{L}} \frac{1}{(|\mathcal{L}|)(|\mathcal{L}| - |\mathcal{A}|)} \sum_{p \in \mathcal{L} \setminus \mathcal{A}} \Delta_p(f) \Delta_p(f^\mathcal{A}).$$

With these notations, and Lemma 2.1, Theorem 2.2 from [1] enables us to get the following result.

Proposition 2.2. *Let Z denote a random variable with a Gaussian distribution with mean zero and variance 1. Let $W = \frac{1}{\sqrt{S}} \sum_{x < n \leq x+y} X(n)$, and let ϕ denote a Lipschitz function satisfying $|\phi(\alpha) - \phi(\beta)| \leq |\alpha - \beta|$ for all real numbers α and β . We have*

$$|\mathbb{E}^\mathcal{F} \phi(W) - \mathbb{E} \phi(Z)| \leq \frac{(\text{Var}^\mathcal{F}(\mathbb{E}^\mathcal{F}(T|X)))^{1/2}}{S} + \frac{1}{2S^{3/2}} \sum_{p \in \mathcal{L}} \mathbb{E}^\mathcal{F} |\Delta_p f|^3.$$

Here conditioning on X means that we are conditioning on the whole vector $(X(n))_{n \geq 1}$. Actually, the bound given by Theorem 2.2 from the paper [1] has $\text{Var}^\mathcal{F}(\mathbb{E}^\mathcal{F}(T|W))$ in the first term instead of $\text{Var}^\mathcal{F}(\mathbb{E}^\mathcal{F}(T|X))$. However, the latter quantity is at least as large as the former because $\mathbb{E}^\mathcal{F}(T|W) = \mathbb{E}^\mathcal{F}(\mathbb{E}^\mathcal{F}(T|X)|W)$ and conditioning reduces variance.

We shall use Proposition 2.2 to estimate $|\mathbb{E}(\phi(W)) - \mathbb{E}(\phi(Z))|$. Note that this quantity is bounded by

$$\mathbb{E} |\mathbb{E}^\mathcal{F}(\phi(W)) - \mathbb{E}(\phi(Z))| \leq \frac{1}{S} \mathbb{E} \left(\text{Var}^\mathcal{F}(\mathbb{E}^\mathcal{F}(T|X))^{1/2} \right) + \frac{1}{2S^{3/2}} \sum_{p \in \mathcal{L}} \mathbb{E} |\Delta_p f|^3.$$

By the Cauchy-Schwarz inequality the first term above is

$$\leq \frac{1}{S} \left(\mathbb{E} \text{Var}^\mathcal{F}(\mathbb{E}^\mathcal{F}(T|X)) \right)^{1/2} \leq \frac{1}{S} \left(\text{Var}(\mathbb{E}^\mathcal{F}(T|X)) \right)^{1/2}.$$

We deduce that

$$(1) \quad |\mathbb{E}(\phi(W)) - \mathbb{E}(\phi(Z))| \leq \frac{1}{S} \left(\text{Var}(\mathbb{E}^\mathcal{F}(T|X)) \right)^{1/2} + \frac{1}{2S^{3/2}} \sum_{p \in \mathcal{L}} \mathbb{E} |\Delta_p f|^3.$$

We will now focus on estimating the two terms in the RHS above. The second term will be estimated in the next section, and the first in Section 4. We now simplify the expression in the first term a little.

For each $p \in \mathcal{L}$, let $\mathcal{N}(p)$ denote all square-free numbers in the interval $[x/p, (x+y)/p]$ which are coprime to p . Note that

$$\Delta_p f = (X(p) - X'(p)) \sum_{k \in \mathcal{N}(p)} X(k),$$

and if $p \in \mathcal{L} \setminus \mathcal{A}$,

$$\Delta_p f^{\mathcal{A}} = (X(p) - X'(p)) \sum_{k \in \mathcal{N}(p)} X^{\mathcal{A}}(k),$$

where $X^{\mathcal{A}}(k)$ is defined in the obvious way replacing X by X' on the primes in \mathcal{A} . Therefore

$$\Delta_p f \Delta_p f^{\mathcal{A}} = (X(p) - X'(p))^2 \left(\sum_{k \in \mathcal{N}(p)} X(k) \right) \left(\sum_{\ell \in \mathcal{N}(p)} X^{\mathcal{A}}(\ell) \right),$$

and since $X(k)$ and $X^{\mathcal{A}}(\ell)$ do not depend on X_p , we see that

$$\begin{aligned} \frac{1}{2} \mathbb{E}^{\mathcal{F}}(\Delta_p f \Delta_p f^{\mathcal{A}} \mid X) &= \left(\sum_{k \in \mathcal{N}(p)} X(k) \right) \left(\sum_{\ell \in \mathcal{N}^{\mathcal{A}}(p)} X(\ell) \right) \\ &= |\mathcal{N}^{\mathcal{A}}(p)| + \sum_{\substack{k \in \mathcal{N}(p), \ell \in \mathcal{N}^{\mathcal{A}}(p) \\ k \neq \ell}} X(k) X(\ell), \end{aligned}$$

where $\mathcal{N}^{\mathcal{A}}(p)$ denotes the set of all square-free integers in $[x/p, (x+y)/p]$ that are not divisible by any prime $q \in \mathcal{A}$. Write the quantity T in Proposition 2.2 as

$$T = \frac{1}{2} \sum_{p \in \mathcal{L}} \sum_{\mathcal{A} \subseteq \mathcal{L} \setminus \{p\}} \nu(\mathcal{A}) \Delta_p f \Delta_p f^{\mathcal{A}},$$

where

$$\nu(\mathcal{A}) := \frac{1}{\binom{|\mathcal{L}|}{|\mathcal{A}|} (|\mathcal{L}| - |\mathcal{A}|)} = \frac{1}{|\mathcal{L}| \binom{|\mathcal{L}|-1}{|\mathcal{A}|}}.$$

Thus,

$$\begin{aligned}\mathbb{E}^{\mathcal{F}}(T \mid X) &= \frac{1}{2} \sum_{p \in \mathcal{L}} \sum_{\mathcal{A} \subseteq \mathcal{L} \setminus \{p\}} \nu(\mathcal{A}) \mathbb{E}^{\mathcal{F}}(\Delta_p f \Delta_p f^{\mathcal{A}} \mid X) \\ &= \sum_{p \in \mathcal{L}} \sum_{\mathcal{A} \subseteq \mathcal{L} \setminus \{p\}} \nu(\mathcal{A}) |\mathcal{N}^{\mathcal{A}}(p)| \\ &\quad + \sum_{p \in \mathcal{L}} \sum_{k \in \mathcal{N}(p)} \sum_{\ell \in \mathcal{N}(p) \setminus \{k\}} \frac{1}{\omega_{\mathcal{L}}(\ell p)} X(k) X(\ell),\end{aligned}$$

where $\omega_{\mathcal{L}}(n)$ denotes the number of distinct prime factors of n that are in \mathcal{L} and the equality above holds because

$$\begin{aligned}\sum_{\mathcal{A} \subseteq \mathcal{L} \setminus \{p\}} \nu(\mathcal{A}) 1_{\{\ell \in \mathcal{N}^{\mathcal{A}}(p)\}} &= \sum_{\mathcal{A} \subseteq \mathcal{L} \setminus (\{p\} \cup \{q|\ell\})} \nu(\mathcal{A}) \\ &= \sum_{k=0}^{|\mathcal{L}| - \omega_{\mathcal{L}}(\ell p)} \frac{1}{|\mathcal{L}| \binom{|\mathcal{L}|-1}{k}} \binom{|\mathcal{L}| - \omega_{\mathcal{L}}(\ell p)}{k} = \frac{1}{\omega_{\mathcal{L}}(\ell p)}.\end{aligned}$$

The last step above involves a combinatorial identity and we leave the pleasure of proving it to the reader; a generalization of this identity appears as Problem B2 of the 1987 Putnam competition see [9]. Now we define

$$T_p := \sum_{k \in \mathcal{N}(p)} \sum_{\ell \in \mathcal{N}(p) \setminus \{k\}} \frac{1}{\omega_{\mathcal{L}}(\ell p)} X(k) X(\ell).$$

Then we may conclude that

$$(2) \quad \text{Var}(\mathbb{E}^{\mathcal{F}}(T \mid X)) = \text{Var}\left(\sum_{p \in \mathcal{L}} T_p\right).$$

3. THE FOURTH MOMENT AND A PARAMETRIZATION OF SOLUTIONS

In this section we shall evaluate the fourth moment

$$\mathbb{E}\left(\sum_{x < n \leq x+y} X(n)\right)^4,$$

for a suitable range of the variables x and y . The techniques involved in this calculation will be used in the proof of our main Theorem. When we expand out the fourth moment, we find that we are counting solutions to the equation

$$n_1 n_2 n_3 n_4 = \square$$

where n_1, n_2, n_3, n_4 are square-free integers with $n_j \in [x, x+y]$ and \square denotes a perfect square. Recall that $y = x\delta$. We begin by parametrizing such solutions.

Write $A = (n_1, n_2)$ and $B = (n_3, n_4)$, and set $n_1 = An_1^*$, $n_2 = An_2^*$, $n_3 = Bn_3^*$ and $n_4 = Bn_4^*$. Then $(n_1^*, n_2^*) = (n_3^*, n_4^*) = 1$ and the equation $n_1 n_2 n_3 n_4 = \square$ is equivalent to $n_1^* n_2^* = n_3^* n_4^*$. Now write $r = (n_1^*, n_3^*)$ and $s = (n_2^*, n_4^*)$. Then $(r, s) = 1$ and we see that $n_1^* = ru$, $n_3^* = rv$, $n_2^* = sv$ and $n_4^* = su$ where u and v are natural numbers with $(u, v) = 1$.

Summarizing the above paragraph, we see that the solutions to $n_1 n_2 n_3 n_4 = \square$ are parametrized by six variables A, B, r, s, u, v , with $(r, s) = (u, v) = 1$ and with

$$n_1 = Aru, n_2 = Asv, n_3 = Brv, n_4 = Bsu.$$

There are additional coprimality conditions to ensure that these numbers are square-free. Since $(1 + \delta)^{-2} \leq n_1 n_2 / (n_3 n_4) \leq (1 + \delta)^2$ we see that

$$(1 + \delta)^{-1} \leq A/B \leq (1 + \delta).$$

Similarly using $n_1 n_3 / (n_2 n_4) = (r/s)^2$ we have

$$(1 + \delta)^{-1} \leq \frac{r}{s} \leq (1 + \delta),$$

and finally using $n_1 n_4 / (n_2 n_3) = u^2 / v^2$ we get that

$$(1 + \delta)^{-1} \leq u/v \leq (1 + \delta).$$

In what follows we shall make use of this parametrization and the above inequalities for the ratios $A/B, r/s, u/v$. One consequence of these inequalities is that if $A \neq B$ then A and B are both $\geq 1/\delta$. Similarly if $r \neq s$ then both r and s are $\geq 1/\delta$ and if $u \neq v$ then u and v are both $\geq 1/\delta$.

Proposition 3.1. *Call any solution to $n_1 n_2 n_3 n_4 = \square$ where the variables are equal in pairs a diagonal solution. The number of non-diagonal solutions to $n_1 n_2 n_3 n_4 = \square$ with $n_j \in [x, x(1 + \delta)]$ and n_j square-free is at most*

$$80x^2 \delta^3 (1 + 2 \log x) (1 + 2\delta \log x).$$

Therefore, with S denoting the number of square-free integers in $[x, x(1 + \delta)]$

$$\mathbb{E} \left(\left(\sum_{k=x}^{x(1+\delta)} X(k) \right)^4 \right) = 3S^2 + O(x^2 \delta^3 (1 + \delta \log x) \log x).$$

Proof. Suppose A, B, r, s, u, v parametrize a non-diagonal solution to $n_1 n_2 n_3 n_4 = \square$. Then either one of u or v is not 1, or one of r or s is not 1; for if $u = v = 1$ and $r = s = 1$ then $n_1 = n_2$ and $n_3 = n_4$. Since these cases are symmetric we will only deal with the case when one of

u or v is not 1, and the total number of solutions is at most twice the number of solutions in this case.

Suppose then that u or v is not 1, and since $(u, v) = 1$ this means that $u \neq v$ and so both u and v are $\geq 1/\delta$. Therefore it follows that $Ar \leq x\delta(1 + \delta)$. Further either $A \neq B$ or $r \neq s$, and so either A or r must be $\geq 1/\delta$. Now suppose A and r are given with $\max(A, r) \geq 1/\delta$ and $Ar \leq x\delta(1 + \delta)$. Since $(1 + \delta)^{-1} \leq A/B \leq (1 + \delta)$ it follows that there are at most $(1 + 2A\delta)$ choices for B . Similarly since $(1 + \delta)^{-1} \leq r/s \leq 1 + \delta$ there are at most $1 + 2r\delta$ choices for s . Finally since $Aru \in [x, x(1 + \delta)]$ there are at most $1 + x\delta/(Ar) \leq 3x\delta/(Ar)$ choices for u , and similarly there are at most $1 + x\delta/(Bs) \leq 3x\delta/(Ar)$ choices for v . Thus the total number of such solutions is

$$\leq 9x^2\delta^2 \sum_{\substack{\max(A, r) \geq 1/\delta \\ Ar \leq x\delta(1 + \delta)}} \frac{(1 + 2A\delta)}{A^2} \frac{(1 + 2r\delta)}{r^2}.$$

This may be bounded by

$$\begin{aligned} &\leq 18x^2\delta^2 \sum_{x\delta \geq A \geq 1/\delta} \frac{1 + 2A\delta}{A^2} \sum_{x\delta \geq r \geq 1} \frac{1 + 2r\delta}{r^2} \\ &\leq 40x^2\delta^3(1 + 2\log x)(1 + 2\delta \log x), \end{aligned}$$

proving our Proposition. \square

When S is of size $x\delta$ (which holds if $x\delta \gg x^{\frac{1}{5}} \log x$), Proposition 3.1 shows that provided $\delta = o(1/\log x)$, the fourth moment matches the fourth moment of a Gaussian.

We now use the ideas of this section to bound the term $\sum_{p \in \mathcal{L}} \mathbb{E}|\Delta_p f|^3$, arising in Proposition 2.2. By the Cauchy-Schwarz inequality we have

$$\mathbb{E}|\Delta_p f|^3 \leq (\mathbb{E}|\Delta_p(f)|^2)^{\frac{1}{2}} (\mathbb{E}|\Delta_p(f)|^4)^{\frac{1}{2}}.$$

As before let $\mathcal{N}(p)$ denote the square-free integers in $(x/p, (x + y)/p]$ which are not multiples of p . Then

$$\mathbb{E}|\Delta_p f|^2 = 2 \sum_{k \in \mathcal{N}(p)} 1 \leq 2 \left(1 + \frac{y}{p}\right).$$

Further we have

$$\mathbb{E}|\Delta_p f|^4 = 8 \sum_{\substack{k_1, k_2, k_3, k_4 \in \mathcal{N}(p) \\ k_1 k_2 k_3 k_4 = \square}} 1,$$

and arguing as in Proposition 3.1 we find that this is $\ll (1 + y/p)^2$ provided $\delta \leq 1/\log x$, where \ll means \leq up to a constant multiple.

Therefore we conclude that

$$\mathbb{E}|\Delta_p(f)|^3 \ll 1 + \left(\frac{y}{p}\right)^{\frac{3}{2}}.$$

Using this estimate for primes $z < p \leq y$ we find that

$$\sum_{z < p \leq y} \mathbb{E}|\Delta_p f|^3 \ll y^{\frac{3}{2}} \sum_{z < p} \frac{1}{p^{\frac{3}{2}}} \ll \frac{y^{\frac{3}{2}}}{z^{\frac{1}{2}}}.$$

If $p > y$ then $\mathbb{E}|\Delta_p f|^3 = 0$ unless there happens to be a square-free multiple of p in $[x, x+y]$ and in this case the expectation is 4. Such primes p must divide $\prod_{x < n \leq x+y} n < (x+y)^y$ and there are at most $y \log(x+y)/\log y$ possibilities for such primes p . We conclude that

$$(3) \quad \sum_{p \in \mathcal{L}} \mathbb{E}|\Delta_p f|^3 \ll \frac{y^{\frac{3}{2}}}{z^{\frac{1}{2}}} + y \frac{\log x}{\log y}.$$

4. PROOF OF THE THEOREM

We now estimate $\text{Var}(\sum_{p \in \mathcal{L}} T_p)$ where we recall that T_p is defined in §2. This quantity equals

$$\sum_{p, q \in \mathcal{L}} \sum_{\substack{k \in \mathcal{N}(p) \\ \ell \in \mathcal{N}(p) \setminus \{k\}}} \sum_{\substack{k' \in \mathcal{N}(q) \\ \ell' \in \mathcal{N}(q) \setminus \{k'\}}} \frac{1}{\omega_{\mathcal{L}}(\ell p) \omega_{\mathcal{L}}(\ell' q)} \mathbb{E}(X(k)X(\ell)X(k')X(\ell')).$$

Above we allow for the possibility that p equals q . The expectation above is 1 exactly when $klk'\ell'$ is a square and zero otherwise. Thus writing $n_1 = kp$, $n_2 = \ell p$, $n_3 = k'q$, $n_4 = \ell'q$ the quantity we seek is

$$\sum_{n_1, n_2, n_3, n_4} \frac{\omega_{\mathcal{L}}((n_1, n_2)) \omega_{\mathcal{L}}((n_3, n_4))}{\omega_{\mathcal{L}}(n_2) \omega_{\mathcal{L}}(n_4)} \leq \sum_{n_1, n_2, n_3, n_4} 1,$$

where $n_j \in [x, x(1+\delta)]$, $n_1 \neq n_2$, $n_3 \neq n_4$, the n_j are square-free with $n_1 n_2 n_3 n_4 = \square$, and (n_1, n_2) and (n_3, n_4) must contain at least one prime factor from \mathcal{L} .

We use the parametrization developed in §3 to estimate this. In the notation used there we find that our quantity above is

$$(4) \quad \leq \sum_{A, B, r, s, u, v} 1.$$

The sum above is over all A, B, r, s, u, v as in our parametrization with the further restraints that $Aru \neq Asv$ and $Brv \neq Bsu$, and that

A and B must each contain at least one prime factor from \mathcal{L} . Our goal is to show that the above quantity is bounded by

$$(5) \quad O\left(x^2\delta^2(1+\delta\log x)\left(\frac{1}{z}+\delta\log x\right)\right).$$

We will obtain this by first fixing A and r and analyzing the restraints on the other variables.

Suppose first that A and r are chosen with $Ar > \delta(x+y)$. If $u \neq v$ then both u and v must be $\geq 1/\delta$ and then we would have $Aru > x+y$. Thus we must have $u = v$ and since $(u, v) = 1$ we have $u = v = 1$. Now $r \neq s$ (else $n_1 = A = n_2$) and so we have that both r and s are at least $1/\delta$. Thus we have $A \ll x\delta$ and $Ar \in [x, x+y]$. Given r the condition $(1+\delta)^{-1} \leq r/s \leq (1+\delta)$ shows that there are $\ll r\delta$ choices for s . Similarly the inequality $(1+\delta)^{-1} \leq A/B \leq (1+\delta)$ shows that given A there are $\ll 1+A\delta$ choices for B . Thus in this case our quantity is

$$\begin{aligned} &\ll \sum_{A \ll x\delta} (1+A\delta) \sum_{x/A \leq r \leq (x+y)/A} r\delta \ll x^2\delta^2 \sum_{A \leq x} \frac{(1+A\delta)}{A^2} \\ &\ll x^2\delta^2 \left(\frac{1}{z} + \delta\log x\right). \end{aligned}$$

The final estimate follows because A must contain at least one prime factor from \mathcal{L} , so that $A \geq z$ and hence $\sum_A 1/A^2 \ll 1/z$.

Now suppose that $Ar < \delta(x+y)$. Recall that either $r = s = 1$ or that both r and s are at least $1/\delta$. We consider these cases separately. In the former case, note that B has $\ll 1+A\delta$ choices, and u and v have at most $x\delta/A$ choices each. Thus this case contributes

$$\ll \sum_{A \leq \delta(x+y)} (1+A\delta)x^2\delta^2/A^2 \ll x^2\delta^2 \left(\frac{1}{z} + \delta\log x\right).$$

Now suppose that we have the second case when $r \geq 1/\delta$. Here there are $\ll 1+A\delta$ choices for B , and given r there are $\ll r\delta$ choices for s . Finally there are $\ll x\delta/(Ar)$ choices for u and $\ll x\delta/(Bs) \ll x\delta/(Ar)$ choices for v . Thus the contribution here is,

$$\begin{aligned} &\ll \sum_{A,r} (1+A\delta)r\delta \frac{x^2\delta^2}{A^2r^2} \\ &\ll x^2\delta^3 \sum_A (1+A\delta)/A^2 \sum_r 1/r \\ &\ll x^2\delta^3 \log x \left(\frac{1}{z} + \delta\log x\right). \end{aligned}$$

Putting all these estimates together gives our bound (5).

Using the bound (5), together with (1), (2) and (3) we conclude that $|\mathbb{E}(\phi(W)) - \mathbb{E}(\phi(Z))|$ is

$$\ll \left(\frac{y}{S}\right)^{\frac{3}{2}} \frac{1}{(\log 1/\delta)^{\frac{1}{2}}} + \frac{y \log x}{S^{\frac{3}{2}} \log y} + \frac{y}{S} (1 + \delta \log x)^{\frac{1}{2}} \left(\frac{1}{\log 1/\delta} + \delta \log x\right)^{\frac{1}{2}}.$$

To deduce the Theorem we combine the above bound with the following simple estimate for $|\mathbb{E}(\phi(W)) - \mathbb{E}(\phi(Z))|$. Since ϕ is Lipschitz we have $|\phi(t) - \phi(0)| \leq |t|$, and so

$$\begin{aligned} |\mathbb{E}(\phi(W)) - \mathbb{E}(\phi(Z))| &\leq |\mathbb{E}(\phi(W) - \phi(0))| + |\mathbb{E}(\phi(Z) - \phi(0))| \\ &\leq \mathbb{E}(|W|) + \mathbb{E}(|Z|) \leq 2. \end{aligned}$$

5. PROOF OF THE COROLLARY

Let ν denote a Gaussian distribution with mean 0 and variance 1, and let μ denote a probability measure. We claim that

$$(6) \quad \mathcal{K}(\mu, \nu) \leq 2\sqrt{\mathcal{W}(\mu, \nu)},$$

and Corollary 1.2 follows as a special case of this estimate.

For any real number t , and a parameter $\epsilon > 0$ consider the function $\Phi^+(\xi; t, \epsilon)$ defined by

$$\Phi^+(\xi; t, \epsilon) = \begin{cases} \epsilon & \text{if } \xi \in (-\infty, t) \\ t + \epsilon - \xi & \text{if } \xi \in [t, t + \epsilon] \\ 0 & \text{if } \xi > t + \epsilon. \end{cases}$$

Note that $\Phi^+(\xi; t, \epsilon)$ is Lipschitz, and moreover $\Phi^+(\xi; t, \epsilon) \geq \epsilon \chi_{(-\infty, t)}(\xi)$. Therefore

$$\begin{aligned} \int_{-\infty}^t d\mu &\leq \frac{1}{\epsilon} \int_{-\infty}^t \Phi^+(\cdot; t, \epsilon) d\mu \leq \frac{1}{\epsilon} \int_{-\infty}^t \Phi^+(\cdot; t, \epsilon) d\nu + \frac{\mathcal{W}(\mu, \nu)}{\epsilon} \\ &\leq \int_{-\infty}^t d\nu + \epsilon + \frac{\mathcal{W}(\mu, \nu)}{\epsilon}. \end{aligned}$$

Choosing $\epsilon = \sqrt{\mathcal{W}(\mu, \nu)}$ we obtain that

$$\int_{-\infty}^t d\mu \leq \int_{-\infty}^t d\nu + 2\sqrt{\mathcal{W}(\mu, \nu)}.$$

An analogous argument, using a similar Lipschitz minorant of the characteristic function of $(-\infty, t)$, gives that

$$\int_{-\infty}^t d\mu \geq \int_{-\infty}^t d\nu - 2\sqrt{\mathcal{W}(\mu, \nu)},$$

and so (6) follows.

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COURANT INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK UNIVERSITY,
251 MERCER STREET, NEW YORK, NY 10012
E-mail address: sourav@cims.nyu.edu

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD, CA
94305.
E-mail address: ksound@stanford.edu